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LETTER TO THE EDITOR

A variational renormalization group approach to two coupled Ising systems

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Abstract. Using the variational block-spin approximation of Kadanoff we construct a remormalization group recursion relation suitable for two-layered Ising systems. We describe the critical behaviour of the Ashkin–Teller model in some detail.

The critical behaviour of two infinite coupled two-dimensional Ising systems is of considerable theoretical interest. The interaction between the systems modifies their separate critical behaviour substantially and in a non-perturbative manner. In this letter we apply a renormalization group block-spin technique (Kadanoff 1975) to describe the critical behaviour of the combined system.

We restrict our considerations to the interactions within the unit cell shown in figure 1. The Hamiltonian of the system then becomes

$$H = -\sum_{\text{sites}} J_1\sigma_i\sigma_j + J_2 s_i s_j + J_{12}\sigma_i s_i + J_4\sigma_i\sigma_j s_i s_j \tag{1}$$

where σ_i, $s_i = \pm 1$ are the two Ising spins at the site i. The following cases are of interest.

(A) $J_{12} = J_4 = 0$. The two systems decouple. There are two critical temperatures (if $J_1 \neq J_2$) and the model is entirely solved in terms of Onsager's solution of the Ising model (Oitmaa and Enting 1975).

(B) $J_{12} = 0$; $J_4 \neq 0$. This model is equivalent to the Ashkin–Teller (AT) model (Fan 1972). A certain amount is known about the shape of the critical surface (see figure 2) (Wu and Lin 1974) by virtue of the dual transformation between the AT and the staggered 8-vertex models (Mittag and Stephen 1971, Wegner 1972). In particular, the

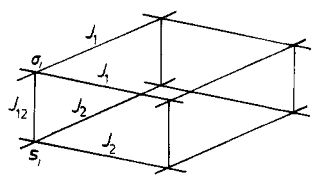

Figure 1. The unit cell of the two-layered Ising system, having nearest-neighbour interactions. J_1 in layer 1, J_2 in layer 2, J_{12} between layers, and a 4-spin interaction, J_4, coupling two nearest-neighbour pairs.

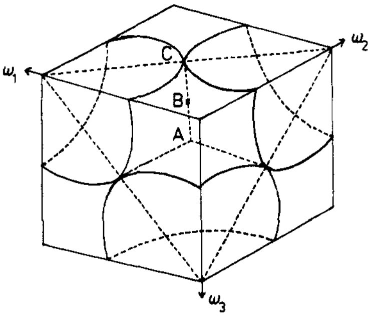

Figure 2. The critical surface of the Ashkin–Teller model, as conjectured by Wu and Lin. The point A represents the fixed point of the Potts model, and B, the fixed point of two decoupled Ising systems. The symmetric fixed point of the 8-vertex model is represented by C.

model with $J_1 = J_2$ coincides with a special case of the exactly soluble 8-vertex model (Baxter 1972). Although the equivalence of the critical exponents cannot be deduced from the duality relation, this does suggest that the AT model may possess continuously varying exponents. (In the language of the renormalization group this requires a line of fixed points.)

(*C*) $J_1 = J_2 = J_4$; $J_{12} = 0$. This particular case of the AT model is the 4-state Potts model (Potts 1952) for which the transition temperature is known but not the critical exponents. This model has an additional symmetry under the transformation $\sigma_i \to \sigma_i s_i$, $s_i \to s_i$.

(*D*) The model with J_{12} as the only non-zero inter-layer coupling has been partially investigated by means of a high temperature expansion by Oitmaa and Enting (1975).

The renormalization group has provided considerable physical insight into critical phenomena (see review by Wilson and Kogut 1974) as well as a rather accurate approximation procedure for obtaining critical exponents. In particular the variational block-spin procedure of Kadanoff (1975) is remarkably simple to implement and has provided exponents which are very close to the exact ones for the ordinary Ising model. We have extended the procedure of Kadanoff to the two-layered system. Our fundamental block is thus the two-layered square of figure 1. We replace each layer of the old lattice (with spins σ_i, s_i respectively) by a new square lattice of Ising spins, μ_i and t_i, having twice the old lattice spacing, in such a manner that the partition function Z, is unchanged

$$Z = \sum_{\{\sigma_i, s_i = \pm 1\}} \exp(-H(\sigma, s))$$

$$= \sum_{\{\mu_i, t_i = \pm 1\}} \exp(-H'(\mu, t))$$

$$= \sum_{\sigma_i s_i \mu_i t_i} \frac{\exp(-H(\sigma, s) + p1\mu_i \Sigma_j \sigma_j + p2 t_i \Sigma_j s_j + p3 \mu_i t_i \Sigma_j \sigma_j s_j)}{e^{a+b+c} + e^{-a-b+c} + e^{a-b-c} + e^{-a+b-c}} \tag{2}$$

where $a = p1\Sigma\sigma_j$; $b = p2\Sigma s_j$; $c = p3\Sigma\sigma_j s_j$ and the sum runs over the four old spins surrounding each of the new spins. $p1$, $p2$, $p3$ are arbitrary parameters. Equation (2) defines the effective Hamiltonian $H'(\mu, t)$ in terms of the Hamiltonian $H(\sigma, s)$, which may be written

$$H(\sigma, s) = -\sum_i K_i S_i(\sigma, s) \tag{3}$$

where the $K_i = J_i/kT$ are a set of coupling constants for the combination of spins $S_i(\sigma, s)$. The structure of equation (2) is designed to ensure that the symmetries of $H(\sigma, s)$ are preserved in $H'(\mu, t)$ in a manifest manner, which will be of paramount importance in subsequent approximations. Although we start with a Hamiltonian containing only 2- and 4-spin terms the application of the renormalization transformation will lead to a new Hamiltonian $H'(\mu, t)$ containing terms of arbitrarily long range. Kadanoff obtains an approximate recursion relation by a procedure that involves dropping couplings between primitive cells of the (σ, s) lattice in such a manner that the resulting approximate free energy is rigorously a lower bound on the exact free energy. Thus in our case the set of symmetric spin functions S_i involve only the eight spins within the unit cell. The thermodynamic properties of the system are determined by the average values of these spin functions, for example, the magnetization is proportional to $\langle \Sigma_i \sigma_i \rangle$. In addition to the usual symmetries we follow Kadanoff in imposing a further permutation symmetry (which is equivalent to performing an initial decimation) which leaves the Hamiltonian invariant under the interchange of any two sites $(\sigma_i, s_i)(\sigma_j, s_j)$ within the unit cell. This then restricts the interaction constants to a set of 34 together with a constant term, K_0.

Successive applications of the transformation lead to a fixed point Hamiltonian, $H^* = H^*(K^*)$. The free parameters $(p1, p2, p3)$ are then fixed by maximizing the free energy as in a usual variational approach. The critical properties of the system are determined by the eigenvalues of the renormalization transformation, linearized about the fixed point, the degree of instability of the point being determined by the number of eigenvalues greater than one. We refer to Kadanoff's paper for a description of the method. More details will be presented in a forthcoming paper. For the moment we describe the main results.

(1) When $J_{12} = J_4 = 0$ we find two phase transitions. Although this case is somewhat trivial we note that we do reproduce the correct crossover exponents with considerable accuracy. Taking the critical temperatures to be $T_{1c} > T_{2c}$ we find several regimes.

(*a*) $T > T_{1c}$. The high temperature fixed point controls this region and there are no relevant eigenvalues (corresponding to the fixed point being stable in all directions).

(*b*) $T = T_{1c}$. This is controlled by a fixed point having relevant temperature and magnetic eigenvalues. (For a description of these see Niemeyer and Van Leeuwen 1974.) However, since $T > T_{2c}$ the eigenvalues corresponding to J_{12} couplings are not relevant.

(*c*) $T_{2c} < T < T_{1c}$. We find a magnetic eigenvalue due to lattice 1 being in an ordered state.

(*d*) $T = T_{2c}$. In this case the fixed point has relevant eigenvalues corresponding to J_{12} couplings between the lattices since the spontaneous magnetization of lattice 1 couples into lattice 2.

We also find crossover exponents ϕ_1, ϕ_2, ϕ_3 corresponding to the critical behaviour of the spin functions

$$S_1 = \sum_{i,j} \sigma_i s_j \qquad S_2 = \sum_{i \neq j,k} \sigma_i \sigma_j s_k \qquad S_3 = \sum_{i \neq j,k} \sigma_k s_i s_j \tag{4}$$

for varying values of the ratio T_{1c}/T_{2c}. Since the two lattices are decoupled it is possible to express these averages as a product of the average values on the two lattices separately:

$$\langle S_1 \rangle = \left\langle \sum_i \sigma_i \right\rangle \left\langle \sum_i s_i \right\rangle \tag{5}$$

and thus their critical behaviour is known. In table 1 we compare the exact and the calculated values of the crossover exponents in the two cases $T_{1c} \neq T_{2c}$ and $T_{1c} = T_{2c}$. For the case where $T_{1c} = T_{2c}$ we also find a marginal eigenvalue ($\lambda = 1$) corresponding to a J_4 coupling, indicating that a line of fixed points exists for models which have a 4-spin coupling (namely the 8-vertex model).

Table 1. The crossover exponents for two decoupled Ising systems.

Exponent	ϕ_1		ϕ_2		ϕ_3	
	Exact	Calculated	Exact	Calculated	Exact	Calculated
$T_{1c} > T_{2c}$	0·125	0·1247	1·0	0·9991	0·125	0·1247
$T_{1c} = T_{2c}$	0·25	0·2494	1·125	1·124	1·125	1·124

(2) $J_{12} = 0$ (Ashkin–Teller model). The easiest way to describe the results is with reference to figure 2, the critical surface conjectured by Wu and Lin (1974). The weights w_i are defined by:

$$w_i = \exp(-\epsilon_i/kT) \tag{6}$$

where

$$\epsilon_1 = 2(J_2 + J_4) \qquad \epsilon_2 = 2(J_1 + J_4) \qquad \epsilon_3 = 2(J_1 + J_2). \tag{7}$$

As T varies from zero to infinity a curve is traced out in the figure from (0, 0, 0) to (1, 1, 1). The whole figure is embedded in the space of all possible couplings—within our approximation, all possible couplings in a unit cell. Under renormalization transformation the system is controlled by fixed points in this space which, in general, have non-zero values of the 6- and 8-spin couplings. When $J_1 = J_2$ the model has two order parameters, the magnetization, $M = \langle \sigma_i \rangle = \langle s_i \rangle$ and the polarization, $P = \langle \sigma_i s_i \rangle$. We denote the corresponding critical exponents by β_m, δ_m and β_e, δ_e respectively, and assume scaling for each set.

When $J_1 \neq J_2$ we confirm that there are two phase transitions. For $J_1 = J_2 < J_4$ our procedure indicates the presence of a line of fixed points giving critical exponents which are continuously dependent on the 4-spin coupling, J_4. Strictly speaking, at a fixed point, our variational approach requires that the free energy is simultaneously a maximum with respect to $p1$ (or $p2$), and $p3$. (By symmetry $p1$ and $p2$ are equivalent when $J_1 = J_2$.) Figure 3 is a plot of J_4 against J_1 along curves for which the free energy is a maximum with respect to $p1$ and $p3$ separately. We note that, while in the centre of the region the curves are clearly distinct, at either end they are so close that a line of fixed points is strongly indicated. Each point on this line is obtained for different values of $p1, p2$ and $p3$. Assuming that the true fixed line lies between the two curves shown in figure 3, then as we move along the line from B, the fixed point of two decoupled Ising systems, to A, the fixed point of the Potts model, the value of the 4-spin coupling increases from 0 at B to $J_1 = J_2$ at A. We find that the exponent δ_m always remains close to 15, while the values of α and of δ_e vary continuously. In table 2 we compare the values of the critical exponents at the two points, A and B. Comparison is also made with the exact values for the decoupled systems (point B) and a high temperature expansion estimate for the Potts model (point A).

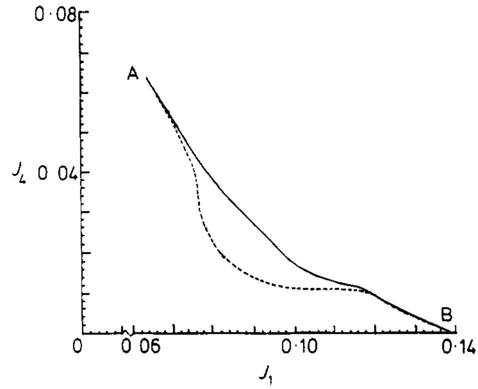

Figure 3. A plot of J_4 against J_1 along curves for which the free energy is a maximum with respect to $p1$ (full curve) and $p3$ (broken curve).

Table 2. The critical exponents obtained from the renormalization group transformation are compared with a high temperature expansion estimate for the Potts model fixed point, A, and with the exact values for the fixed point of two decoupled Ising systems, B.

	A		B	
Critical Exponent	Calculated	High temperature estimate	Calculated	Exact value
α	0·5009	0·6±0·1	0·0017	0
δ_m	15·65	15·8±0·8	15·040	15
δ_e	15·65	15·8±0·8	7·021	7
β_m	0·09005	0·089±0·03	0·1246	0·125
β_e	0·09005	0·089±0·03	0·2494	0·25

The value of α in the Potts limit ($J_1 = J_2 = J_4$) is strikingly close to the value $\alpha_{\text{Potts}} = \frac{1}{2}$. The 8-vertex model has the same value of α at its symmetric point ($K^+ = K^- = \lambda$ in the notation of Kadanoff and Wegner 1971), and, under the dual transformation this point transforms into the point labelled C (figure 2). For the Ashkin–Teller model the calculation of α at C ($J_1 = J_2 = -J_4 = \infty$) involves the use of negative $p3$ for which it has not yet been possible to obtain any exact results. However, α does appear to become negative as this point is approached from B, and may tend to $\alpha = -\infty$ at B. This is the same as the 8-vertex value at the point $K^+ = K^- = -\lambda = \infty$ which is mapped into A in figure 2. There does, therefore, seem to be a reciprocal relationship between the specific heat exponents in the two models.

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